Problem 14

Evaluate $\int \sqrt{\tan x}\, dx$.

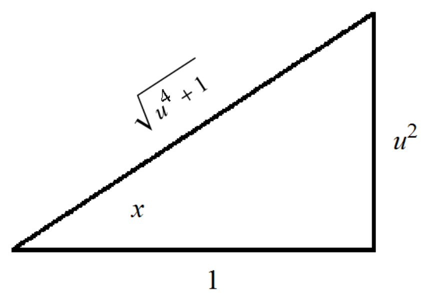
Solution

Start off by making the u-substitution,

$$u = \sqrt{\tan x} \qquad \rightarrow \qquad u^2 = \tan x$$

$$du = \frac{1}{2}(\tan x)^{-\frac{1}{2}} \cdot \sec^2 x\, dx = \frac{\sec^2 x}{2\sqrt{\tan x}}\, dx \qquad \rightarrow \qquad dx = \frac{2\sqrt{\tan x}}{\sec^2 x}\, du.$$

A right triangle can be set up to determine $\sec x$ using $\tan x = u^2$. We see that



$$\sec x = \sqrt{u^4 + 1} \quad \rightarrow \quad \sec^2 x = u^4 + 1.$$

Hence,

$$\int \sqrt{\tan x}\, dx = \int u \cdot \frac{2u}{u^4 + 1}\, du.$$

The integral to solve is therefore

$$\int \frac{2u^2}{u^4 + 1}\, du.$$

It is important to note here that the denominator can be factored. If we square $u^2 + 1$, we get $u^4 + 2u^2 + 1$. To get rid of the cross term, $2u^2$, we use a product of conjugates. That is,

$$u^4 + 1 = [(u^2 + 1) - \sqrt{2}u][(u^2 + 1) + \sqrt{2}u].$$

The integral becomes

$$\int \frac{2u^2}{(u^2 + 1 - \sqrt{2}u)(u^2 + 1 + \sqrt{2}u)}\, du.$$

Our task now is to break up the integrand using partial fraction decomposition. As the factors in the denominator are quadratic in u, we will have the following decomposition.

$$\frac{2u^2}{(u^2 + 1 - \sqrt{2}u)(u^2 + 1 + \sqrt{2}u)} = \frac{Au + B}{u^2 + 1 - \sqrt{2}u} + \frac{Cu + D}{u^2 + 1 + \sqrt{2}u}$$

Multiply both sides by the LCD.

$$2u^2 = (Au + B)(u^2 + 1 + \sqrt{2}u) + (Cu + D)(u^2 + 1 - \sqrt{2}u)$$

Since we have four unknowns to determine, choose 4 random values for u to get four equations.

$$\begin{aligned} u = 0: &\quad 0 = B + D \\ u = 1: &\quad 2 = (A + B)(2 + \sqrt{2}) + (C + D)(2 - \sqrt{2}) \\ u = 2: &\quad 8 = (2A + B)(5 + 2\sqrt{2}) + (2C + D)(5 - 2\sqrt{2}) \\ u = 3: &\quad 18 = (3A + B)(10 + 3\sqrt{2}) + (3C + D)(10 - 3\sqrt{2}) \end{aligned}$$

Solving the system with substitution yields

$$A = \frac{1}{\sqrt{2}}, \quad B = 0, \quad C = -\frac{1}{\sqrt{2}}, \quad D = 0.$$

Thus,

$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} \int \frac{u}{u^2 - \sqrt{2}u + 1}\, du - \frac{1}{\sqrt{2}} \int \frac{u}{u^2 + \sqrt{2}u + 1}\, du.$$

Complete the square in the denominators.

$$\begin{aligned} \int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} \int & \frac{u}{u^2 - \sqrt{2}u + \left(\frac{\sqrt{2}}{2}\right)^2 + 1 - \left(\frac{\sqrt{2}}{2}\right)^2}\, du \\ & - \frac{1}{\sqrt{2}} \int \frac{u}{u^2 + \sqrt{2}u + \left(\frac{\sqrt{2}}{2}\right)^2 + 1 - \left(\frac{\sqrt{2}}{2}\right)^2}\, du. \end{aligned}$$

The integrands become

$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} \int \frac{u}{\left(u - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}}\, du - \frac{1}{\sqrt{2}} \int \frac{u}{\left(u + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}}\, du.$$

The point of completing the square is so that we can now solve the integrals by trigonometric substitution. Let

$$\begin{aligned} u - \frac{\sqrt{2}}{2} &= \sqrt{\frac{1}{2}} \tan\theta & \qquad u + \frac{\sqrt{2}}{2} &= \sqrt{\frac{1}{2}} \tan\alpha \\ du &= \frac{1}{\sqrt{2}} \sec^2\theta\, d\theta & \qquad du &= \frac{1}{\sqrt{2}} \sec^2\alpha\, d\alpha \end{aligned}$$

so that

$$\begin{aligned} \left(u - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2} &= \left(\sqrt{\frac{1}{2}} \tan\theta\right)^2 + \frac{1}{2} = \frac{1}{2}\tan^2\theta + \frac{1}{2} = \frac{1}{2}(\tan^2\theta + 1) = \frac{1}{2}\sec^2\theta \\ \left(u + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2} &= \left(\sqrt{\frac{1}{2}} \tan\alpha\right)^2 + \frac{1}{2} = \frac{1}{2}\tan^2\alpha + \frac{1}{2} = \frac{1}{2}(\tan^2\alpha + 1) = \frac{1}{2}\sec^2\alpha. \end{aligned}$$

Plug these expressions into the integrals now.

$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} \int \frac{\frac{1}{\sqrt{2}}\tan\theta + \frac{1}{\sqrt{2}}}{\frac{1}{2}\sec^2\theta} \cdot \frac{1}{\sqrt{2}} \sec^2\theta\, d\theta - \frac{1}{\sqrt{2}} \int \frac{\frac{1}{\sqrt{2}}\tan\alpha - \frac{1}{\sqrt{2}}}{\frac{1}{2}\sec^2\alpha} \cdot \frac{1}{\sqrt{2}} \sec^2\alpha\, d\alpha$$

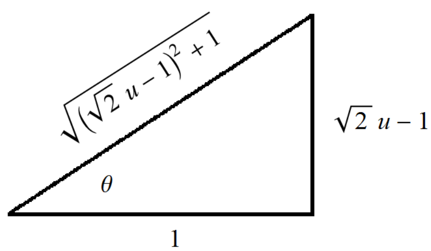
Simplify the integrands.

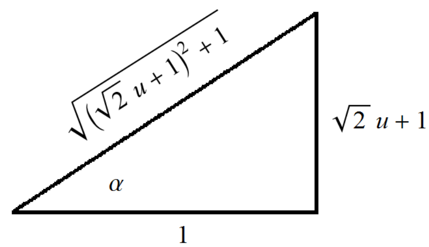
$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} \int (\tan\theta + 1)\, d\theta - \frac{1}{\sqrt{2}} \int (\tan\alpha - 1)\, d\alpha$$

Finally we can integrate.

$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} (\ln|\sec\theta| + \theta) - \frac{1}{\sqrt{2}} (\ln|\sec\alpha| - \alpha) + E,$$

where E is an arbitrary constant of integration. Now that the integration is done, we have to return to the original integration variable, x. Since $\sqrt{2}u - 1 = \tan\theta$ and $\sqrt{2}u + 1 = \tan\alpha$, we can write expressions for θ and α in terms of u and we can draw right triangles to determine $\sec\theta$ and $\sec\alpha$. We have





$$\theta = \tan^{-1}(\sqrt{2}u - 1) \qquad \alpha = \tan^{-1}(\sqrt{2}u + 1)$$

$$\sec\theta = \sqrt{(\sqrt{2}u - 1)^2 + 1} \qquad \sec\alpha = \sqrt{(\sqrt{2}u + 1)^2 + 1},$$

so we get

$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} \left[\ln\sqrt{(\sqrt{2}u - 1)^2 + 1} + \tan^{-1}(\sqrt{2}u - 1)\right] - \frac{1}{\sqrt{2}} \left[\ln\sqrt{(\sqrt{2}u + 1)^2 + 1} - \tan^{-1}(\sqrt{2}u + 1)\right] + E.$$

Combine the terms into one by factoring.

$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} \left[\ln\sqrt{(\sqrt{2}u - 1)^2 + 1} + \tan^{-1}(\sqrt{2}u - 1) - \ln\sqrt{(\sqrt{2}u + 1)^2 + 1} + \tan^{-1}(\sqrt{2}u + 1)\right] + E$$

Combine the logarithms into one, bring the exponent of $1/2$ in front, and plug in $u = \sqrt{\tan x}$ to get the final answer.

$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}} \left[\frac{1}{2} \ln \frac{(\sqrt{2\tan x} - 1)^2 + 1}{(\sqrt{2\tan x} + 1)^2 + 1} + \tan^{-1}(\sqrt{2\tan x} - 1) + \tan^{-1}(\sqrt{2\tan x} + 1)\right] + E$$